

# On the thermodynamics of scale factor dual Universes

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**Abstract** The thermodynamical aspects of the conformal time scale factor duality (SFD) of cosmological models within Einstein Gravity are investigated. We derive the SFD transformations of the thermodynamical quantities describing the thermal evolution of the matter fluid and of the apparent horizon. The thermodynamical properties of the self-dual cosmological models with a modified Chaplygin gas are studied in detail. We deduce the restrictions on the equation of state parameters that allow to extend scale factor duality as a UV/IR symmetry of the cosmological models consistent with their thermodynamical behavior.

**Keywords** Scale Factor Duality · UV/IR Symmetries · Modified Chaplygin Gas Self-Dual Thermodynamics

## 1 Introduction

Although the effective field theories used in the description of the universe evolution contain a few distinct energy (or length) scales and other dimensionful parameters, it is expected that local conformal transformations take place as an *asymptotic* symmetry [1, 2, 3, 4, 5, 6]. There are a variety of options for dynamical and spontaneous breaking of the conformal (Weyl) symmetry which lead to rather realistic cosmological models, consistent with recent astrophysical observations [7].

The *scale factor duality* (SFD) invariant cosmologies [8, 9, 10, 11, 12, 13], for Friedmann-Robertson-Walker

(FRW) universes<sup>1</sup>

$$ds^2 = a^2(\eta) \left( -d\eta^2 + \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right), \quad (1)$$

provide examples of models with a manifest residual  $Z_2$  symmetry, where the Weyl transformations are partially broken while preserving the invariance under the subgroup of scale factor inversions,  $a \mapsto 1/a$ . They represent a symmetry of the space of solutions of the Friedmann equations,

$$\begin{aligned} \rho' + 3\frac{a'}{a}(\rho + p) &= 0, \quad \frac{\kappa^2}{6}\rho = \left(\frac{a'}{a^2}\right)^2 + \frac{k}{a^2}, \\ \frac{\kappa^2}{2}p &= \left(\frac{a'}{a^2}\right)^2 - \frac{2a''}{a^3} - \frac{k}{a^2}, \end{aligned} \quad (2)$$

interchanging small and large physical scales. When combined with time reflections, they give rise to scale factor duality transformations known to be an important tool in the construction of pre-big-bang cosmological models [8, 9]. In the case when the SFD requirement is implemented in *conformal time*, combined with the specific transformation of the matter fluid with equation of state (EoS)  $p/\rho = \omega(\rho)$  [13, 14],

$$\begin{aligned} \tilde{a}(\tilde{\eta}) &= c_0^2/a(\eta), \quad \tilde{\eta} = \pm\eta + \text{const.}, \\ \tilde{a}^2\tilde{\rho}(\tilde{a}) &= a^2\rho(a), \\ \tilde{a}^2[3\tilde{p}(\tilde{a}) + \tilde{\rho}(\tilde{a})] &= -a^2[3p(a) + \rho(a)], \\ \omega(\rho) + \tilde{\omega}(\tilde{\rho}) &= -\frac{2}{3}, \end{aligned} \quad (3)$$

one observes that, apart from keeping invariant the form of the Friedmann equations (2), they also manifest an interesting *UV/IR* feature. Namely, for fluids satisfying the null energy condition (NEC), i.e. when

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<sup>1</sup>We denote  $\kappa^2 \equiv 16\pi G$ ,  $f' \equiv \frac{df}{d\eta}$  a derivative with respect to conformal time  $\eta$ , and  $k = 0, \pm 1$  for spatially flat, open or closed universes.

$-1 \leq \omega \leq \frac{1}{3}$  (and the same for its dual  $\tilde{\omega}$ ), *high energy* densities  $\rho$  and large values of the scalar Ricci curvature are always mapped into *small* corresponding values in the dual models. We shall consider universes with one period of decelerated and one period of accelerated expansion, such that the “conformal lifetime” is finite; say  $0 \leq |\eta| \leq \eta_f$  as  $0 \leq a(\eta) \leq \infty$ . Then, as we have recently demonstrated [14], there are a few distinct realizations of the conformal time SFD (3), depending on the particular choice of the conformal time transformations: (i)  $\tilde{\eta} = -\eta$  gives rise to an expansion/expansion type of pre-big-bang and conformal cyclic cosmologies; (ii)  $\tilde{\eta} = \eta + \eta_f$  leads to contraction/expansion pre-big-bang models and (iii)  $\tilde{\eta} = 2\eta_c - \eta$  defines a family of SFD self-dual post-big-bang cosmologies [13], with  $\eta_c = \eta_f/2$  being the instant of deceleration/acceleration transition, i.e.  $\ddot{a}(\eta_c) = 0$ . The investigation of the consequences of the *self-duality* requirement [13] has revealed a new application of SFD out of the scope of the original pre-big-bang scenario [8,9]—it allows one to describe the short distance and early time (UV) behavior of the universe evolution in terms of its large distance and late times (IR) behavior.

The present paper is devoted to the investigation of the thermodynamical (TD) properties of the self-dual cosmological models introduced in ref.[13]. The problem we address here concerns the conditions that permit one to extend the UV/IR symmetry of the curvature and energy density to the thermodynamical characteristics of the *SFD symmetric universes* as well. More precisely, our aim is to derive the SFD transformations of the temperature, entropy, internal energy, etc., and to further select the values of the equation of state parameters of these models such that high temperatures are transformed into low temperatures and vice-versa. As expected, the self-duality of the FRW equations also allows us to demonstrate the self-duality of the thermodynamical description of the corresponding pairs of *dual apparent horizons* within the framework of the Cai-Kim prescription for their temperature and entropy [15]. It deserves to highlight one interesting result of our study of the thermodynamical features of the simplest *self-dual fluids*, based on the radiation-like Chaplygin gas with EoS

$$p = \frac{1}{3}\rho - \frac{4}{3}\rho^\delta \Lambda^{\delta-1}$$

namely that within the interval  $\delta \in [\frac{3}{4}, 1]$  not only the curvature and matter density, but also the temperature and the pressure turn out to be monotonically decreasing with the increasing of the scale factor. As a consequence the *UV/IR symmetric* “thermal history” of these self-dual cosmological models appears to be quite similar to the standard  $\Lambda$ CDM model.

## 2 Thermodynamical aspects of scale factor duality

Given the conformal time SFD transformation laws (3), we are interested in deriving the transformations of all the remaining thermodynamical characteristics of a barotropic fluid within a constant comoving volume  $V_0$ . The thermodynamical behavior of such fluids is described by an adiabatic process in thermal equilibrium (see, e.g. [16]), with the entropy, energy and pressure densities  $s$ ,  $\rho$  and  $p$ , respectively, all functions of temperature  $T$  only. Then as a consequence of the first TD law,  $dE = TdS - pdV$ , one gets the well known expressions

$$s = \frac{p + \rho}{T}, \quad \frac{ds}{dT} = \frac{1}{T} \frac{d\rho}{dT}, \quad \frac{d\rho}{dT} = \frac{1}{T} (p + \rho) \frac{dp}{dT}. \quad (4)$$

for the total internal energy  $E \equiv \rho V$ , the entropy  $S \equiv sV$  and the physical volume  $V = V_0 a^3$ . Since the entropy is time independent  $dS/dt = 0$ , the entropy density scales as  $s = S/a^3$  for a constant  $S$ . From the simple form of the above identities, one may easily derive the SFD transformations of temperature, entropy and internal energy:

$$\frac{\tilde{E}}{\tilde{a}} = \frac{E}{a}, \quad \tilde{S} = S, \quad \frac{1}{\tilde{a}} \left( \tilde{S} \tilde{T} - \frac{2}{3} \tilde{E} \right) = -\frac{1}{a} \left( ST - \frac{2}{3} E \right), \quad (5)$$

by taking into account Eq.(3). This establishes the desired SFD relations between the thermodynamics of a pair of *dual* fluids.

### 2.1 Self-Dual Fluids Thermodynamics

According to the results of ref.[13], by taking a pair of scale factor dual FRW solutions  $(a, \rho, p)$  and  $(\tilde{a}, \tilde{\rho}, \tilde{p})$  and requiring their *self-duality*:

$$\tilde{\rho}(\tilde{a}) = \rho(\tilde{a}), \quad \tilde{p}(\tilde{a}) = p(\tilde{a}), \quad \tilde{\omega}(\tilde{\rho}) = \omega(\tilde{\rho}), \quad \tilde{\eta} = 2\eta_c - \eta, \quad (6)$$

one may obtain a special class of (post-big-bang) cosmologies in which SFD is a symmetry of one *single* FRW universe with two periods of acceleration. That is, SFD maps accelerated into decelerated phase—thus the early universe into the late universe—, through a reflection about the SFD-invariant instant  $\eta_c$ .

The symmetry requirements (6) impose quite strong restrictions on the density of a self-dual fluid. Combining (6) and (3) we find that its EoS must obey

$$\frac{c_0^2}{a(\eta)} = a(2\eta_c - \eta), \quad \rho(\Omega a) = \Omega^{-2} \rho(a),$$

$$\omega(\rho) + \omega(\Omega^{-2} \rho) = -\frac{2}{3}, \quad \Omega \equiv \frac{c_0^2}{a^2}. \quad (7)$$

Within a vast family of two-component interacting fluids, the “radiation-like” modified Chaplygin gas models [17, 18, 19],

$$\rho = \left( \rho_A^\delta + \rho_r^\delta a^{-4\delta} \right)^{\frac{1}{\delta}}, \quad p = \frac{1}{3}\rho - \frac{4\rho_A^\delta}{3} \rho^{1-\delta}, \quad (8)$$

are the only ones to satisfy these constraints, as long as  $\tilde{\rho}_r = \rho_r = c_0^4 \rho_A$  and  $\tilde{\rho}_A = \rho_A$ . In the case of  $\delta > 0$ , with  $\rho_A \leq \rho < \infty$ , they provide a class of self-dual,  $\Lambda$ CDM-like big-bang cosmologies [13]. According to Eq.(8), the self-dual fluid behaves as radiation when  $a \rightarrow 0$ , and as a cosmological constant  $\rho_A$  when  $a \rightarrow \infty$ . Thus the big-bang singularity is followed by a decelerated and then an accelerated period of expansion within a finite conformal lifetime whose duration is given by (see ref.[13])

$$\eta_f = 2\eta_c = \frac{\sqrt{3}}{4\delta (\rho_r \rho_A)^{\frac{1}{4}}} \frac{\left[ \Gamma\left(\frac{1}{4\delta}\right) \right]^2}{\Gamma\left(\frac{1}{2\delta}\right)}.$$

The transition between the deceleration and acceleration epochs occurs at the instant  $\eta_c$ , corresponding to the SFD invariant value of the scale factor  $a(\eta_c) = c_0 = (\rho_r/\rho_A)^{\frac{1}{4}}$ , when  $\rho_0 = \rho(\eta_c) = 2\rho_A$  and the deceleration parameter  $q(a) \equiv \frac{1}{2} \left( 1 + 3\frac{p}{\rho} \right)$  vanishes. We should mention the rather different nature of the solutions with  $\delta < 0$  — their curvature is monotonically decreasing and bounded,  $R_{dS} \geq R \geq 0$ . They describe *eternally* expanding *non-singular* universes, which are asymptotically de Sitter at the remote past  $\eta \rightarrow -\infty$  and accelerated up to the moment  $\eta_c$ . Their late time decelerated phase is dominated by radiation in the far future,  $\eta \rightarrow \infty$ .

We next consider the thermodynamics of the self-dual fluids (8). The evolution of their temperature  $T(a)$  as a function of the scale factor or, equivalently, its dependence on the physical volume  $V = V_0 a^3$ , can be easily derived from the last of the Eqs.(4), once the EoS of the fluid is given:

$$4\sigma_{sb} T^4 = \rho \left( 1 - \frac{\rho_A^\delta}{\rho^\delta} \right)^{4-\frac{3}{\delta}} = \frac{\rho_r}{a^4} \left( 1 + \frac{\rho_A^\delta}{\rho_r^\delta} a^{4\delta} \right)^{\frac{4}{\delta}-4},$$

$$\sigma_{sb} = \frac{\pi^2 K_B^4}{60\hbar^3}, \quad (9)$$

where we have fixed the constant of integration to be equal to the Stephan-Boltzmann constant  $\sigma_{sb}$ . Such a choice reproduces, in the limit  $\rho_A \rightarrow 0$ ,<sup>2</sup> the Stephan-Boltzmann law  $\rho = 4\sigma_{sb} T^4$  for an ultrarelativistic gas. Notice, however, that the thermal evolution of the scale factor  $a(T)$  and of the fluid density  $\rho(T)$  can be explicitly found only in a few cases (say for  $\delta = 1$  and  $\delta = 3/4$ ), when the inversion of Eqs.(9) is available.

<sup>2</sup>I.e. when (8) reduces to pure radiation, with  $p = \frac{1}{3}\rho$  and  $\rho = \rho_r/a^4$ .

Nevertheless, the above equations will allow us to deduce their TD properties in general, and also to establish the restrictions on the values of EoS parameter  $\delta$  that ensure their physical consistency.

Let us remind one important feature of the equilibrium processes (in homogeneous systems), namely that all the thermodynamical potentials—the internal energy  $E(S, V)$ , the free energy  $F(T, V) = E - TS$ , the Gibbs potential  $\Phi(T, P) = E + pV - ST$ , etc.—have to remain at their minimal values. The conditions to guarantee the stability of these minima, say, against thermal or pressure fluctuations, are given by the so called *thermodynamic inequalities* (see for example [20]). In the case of the adiabatic equilibrium processes occurring in the self-dual fluids (8) it is sufficient to impose that its adiabatic compressibility is always positive, i.e.

$$K_S = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_S > 0, \text{ or equivalently } \frac{dp}{d\rho} > 0.$$

In order to establish the consequences of such a “TD stability restriction” we have to examine the behavior of the temperature  $T(\rho)$  and the pressure  $p(\rho)$  as functions of the fluid density. Observe that *they are not monotonic functions* for all the values of  $\delta$ ,<sup>3</sup> due to the existence of real zeros  $\rho_{cr}(\delta) = (4 - 4\delta)^{\frac{1}{\delta}} \rho_A$  of the derivatives  $\frac{dp}{d\rho}$  and  $\frac{dT}{d\rho}$ , placed within the intervals  $\rho_{cr} \in (\infty, \rho_A]$  when  $0 < \delta \leq 3/4$ , or  $\rho_{cr} \in [\rho_A, 0]$  for  $\delta < 0$ . The corresponding “critical values”

$$p_{cr}(\delta) = -\frac{4}{3}\delta\rho_A(4 - 4\delta)^{\frac{1-\delta}{\delta}},$$

$$T_{cr} = T_* (4 - 4\delta)^{\frac{1-\delta}{\delta}} (3 - 4\delta)^{\frac{4\delta-3}{4\delta}},$$

$$T_* \equiv \left( \frac{\rho_A}{4\sigma_{sb}} \right)^{\frac{1}{4}} \quad (10)$$

represent the maxima of  $p(\rho)$  and  $T(\rho)$  for  $\delta < 0$  and the minima for  $0 < \delta < 3/4$ . Notice that the  $\delta = 3/4$  model admits a minimum for the pressure only. One can also easily derive the specific *scaling* behavior of all the TD quantities around these critical points (for  $\delta \leq 3/4$ ), viz.

$$K_S \sim |T - T_{cr}|^{-\frac{1}{2}},$$

$$\rho - \rho_{cr} \sim |T - T_{cr}|^{\frac{1}{2}},$$

$$p - p_{cr} \sim (\rho - \rho_{cr})^2, \quad (11)$$

etc., with universal critical exponents independent of the values of  $\delta$ . Although the second derivatives of the TD potentials manifest power-like singularities, typical for second order phase transitions, we should emphasize however that the above described critical behavior

<sup>3</sup>Let us remind, however, that  $\rho(a)$  is always monotonically decreasing for all  $a \in (0, \infty)$ .

does not satisfy all the requirements [20] needed for the realization of such phase transitions.

The above discussion makes it clear that the fluids with

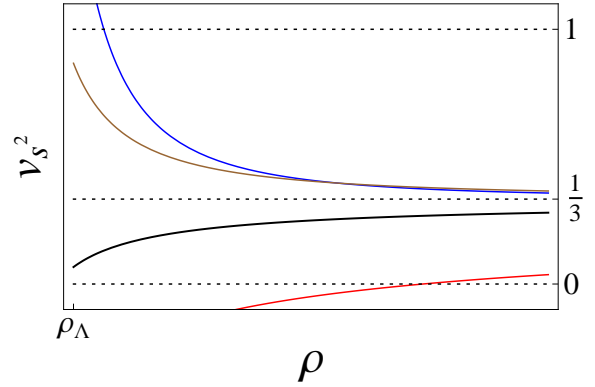
$$\begin{aligned} (i) \quad & \delta < 0, \quad (ii) \quad 0 < \delta < 3/4, \\ (iii) \quad & \delta = 3/4, \quad (iv) \quad \delta > 3/4 \end{aligned} \quad (12)$$

should represent different thermodynamical properties. In case (ii) we realize that  $K_S$  changes its sign at  $\rho_{cr}$  and the Chaplygin-like fluids, filling the expanding asymptotically de Sitter universe, manifest a rather *unphysical TD behavior*—after reaching its minimal value  $T_{cr}$ , the temperature of the expanding fluid starts *increasing* towards infinity, so the fluid becomes “extremely rarefied and infinitely hot”.

The thermal history of the eternal universe in case (i), when  $\delta < 0$ , has a cooler, much more reasonable, but still *unstable* behavior, since again the TD inequality  $K_S > 0$  is not respected during all of its evolution. Both the initial de Sitter state, and the final radiation stage have a vanishing temperature. The transition between these two cold phases represents a “heating-to-cooling” process, with the temperature increasing from zero up to  $T_{cr}(\delta < 0) = T_{max}$  before decreasing to zero again. It is worth reminding that for small enough values of  $|\delta|$  this models describe a hilltop inflationary period (in the neighborhood of initial de Sitter state), that turns out to fit quite well with the Planck-2015 data [7], cf. ref.[13].

The models with  $\delta > \frac{3}{4}$  are examples of stable TD behavior, since  $K_S$  remains always positive, and their TD evolution is described by monotonically decreasing temperature  $T(\rho) \in (\infty, 0)$  and pressure  $p \in (\infty, -\rho_\Lambda)$ , as in most of the  $\Lambda$ CDM-like models. We shall further impose a natural upper bound  $\delta \leq 1$  that excludes the models whose (Ricci) curvature  $R = 2\rho_\Lambda^\delta (\rho)^{1-\delta}$  increases with the decreasing of the fluid density.

Finally, the thermodynamics of the  $\delta = 3/4$  model (iii) is quite different from all the other cases we have described above. Here,  $\rho_{cr} = \rho_\Lambda$  is *not* an extremum of the temperature, which is a monotonic function. So this is the only asymptotically de Sitter self-dual model with a *non-vanishing* final temperature  $T_*$ , which nevertheless shares (together with the simplest  $\delta = 1$  model) the standard radiation-like Stephan-Boltzmann law  $\rho = 4\sigma_{sb}T^4$ . The pressure reaches its minimum  $p_{cr} = -\rho_\Lambda$  only at the final de Sitter boundary, when  $a \rightarrow \infty$  and  $\rho \rightarrow \rho_{cr} = \rho_\Lambda$ . It is worthwhile to mention that such  $\delta = \frac{3}{4}$  self-dual universe exhibits many of the features of the Chaplygin-like *quintessence* models [21, 17, 18]. Certain similarities with the standard  $\Lambda$ CDM cosmology should be highlighted as well [13]—its energy density at early times is approximated by radiation,  $\rho \approx \rho_r/a^4$ ,



**Fig. 1** Behavior of the speed of sound for different values of  $\delta$ . Blue:  $\delta > \frac{3}{2}$ ; Brown:  $1 < \delta < \frac{3}{2}$ ; Black:  $\frac{3}{4} < \delta < 1$ ; Red:  $0 < \delta < \frac{3}{4}$ .

while at relatively late times it behaves as a cosmological constant with cold dark matter, i.e.  $\rho \approx \rho_\Lambda + \rho_{dm}/a^3$ .

Another set of restrictions on the values of the parameter  $\delta$  arises from the requirement that the speed of sound  $v_s^2 = \frac{dp}{d\rho}$  should be smaller than  $c^2 = 1$ , i.e.  $0 \leq v_s^2 \leq 1$ , which imposes

$$0 \leq v_s^2 = \frac{1}{3} - \frac{4(1-\delta)}{3} \frac{\rho_\Lambda^\delta}{\rho^\delta} \leq 1 \quad (13)$$

for the modified Chaplygin gas models. Since  $v_s^2(\rho)$  turns out to be a monotonic function, it is enough to consider the consequences of Eq.(13) for its initial and final values  $v_{s(in)}^2 \equiv v_s^2(a=0)$  and  $v_{s(f)}^2 \equiv v_s^2(a \rightarrow \infty)$ . They are

$$\begin{aligned} v_{s(in)}^2 &= -1 - \frac{4}{3}|\delta| < 0, \quad v_{s(f)}^2 = \frac{1}{3}, \quad \text{for } \delta < 0, \\ v_{s(in)}^2 &= \frac{1}{3}, \quad v_{s(f)}^2 = -1 + \frac{4}{3}\delta, \quad \text{for } \delta > 0. \end{aligned} \quad (14)$$

This makes it evident which are the selected ranges of values of  $\delta$  that provide *hydrodynamically consistent* equations of state:

- Models with  $\delta > \frac{3}{2}$  admit sound waves with superluminal velocity  $v_s^2 > 1$  (blue line in Fig.1), and hence must be discarded.

- The only self-dual fluids that satisfy (13) are those with  $\delta \in [\frac{3}{4}, \frac{3}{2}]$ . Notice, however, that at  $\delta = 1$  there occurs an important change in the behavior of  $v_s(\rho)$ . We have

$$\begin{aligned} \frac{1}{3} &\geq v_s^2(\rho) \geq v_s^2(\rho_\Lambda) \geq 0 & \text{for } \frac{3}{4} \leq \delta \leq 1; \\ \frac{1}{3} &< v_s^2(\rho) \leq v_s^2(\rho_\Lambda) \leq 1 & \text{for } 1 < \delta \leq \frac{3}{2}. \end{aligned}$$

(Black and brown lines in Fig.1, respectively.) While, for  $\delta < 1$ ,  $v_s^2$  is a monotonic function decreasing with the density, for  $\delta > 1$  it becomes a function which *increases* as the density decreases. Note that the latter is a rather *unphysical* behavior.

• For all the cosmological models with  $\delta < \frac{3}{4}$ , the lower bound is *not respected*, i.e. we have  $v_s^2 < 0$  for a certain range of values of the fluid density (red line in Fig.1), which causes thermo- and hydrodynamical instabilities when the zero sound velocity limit is violated.

As we have already mentioned, the family of models with  $\delta < \frac{3}{4}$  is in fact separated in two distinct classes — one for  $\delta < 0$  and the other for  $\delta \in (0, \frac{3}{4})$  — with quite different geometrical and thermodynamical properties. These differences are highlighted in the description of the fluid as a self-interacting scalar field. The equation of state (8) is equivalent to the following scalar field potential [13]

$$V(\sigma) = \frac{2}{\kappa^2 L^2} \left\{ \left[ \cosh^2 \left( \frac{\delta}{\sqrt{2}} \kappa \sigma \right) \right]^{\frac{1}{\delta}} + 2 \left[ \cosh^2 \left( \frac{\delta}{\sqrt{2}} \kappa \sigma \right) \right]^{\frac{1-\delta}{\delta}} \right\}, \quad L^2 \equiv \frac{2}{\rho_\Lambda}. \quad (15)$$

As one can easily verify, the relation between the EoS parameter  $\delta$  and the scalar field mass  $m_\sigma^2 = V''(0)$  at the “de Sitter extremum”  $\sigma = 0$  of the potential  $V(\sigma)$ ,

$$m_\sigma^2 L^2 = 2\delta(3 - 2\delta), \quad (16)$$

provides a (partial) explanation of the qualitatively different behavior of the models corresponding to the discussed different ranges of values for  $\delta$ . For example, in the cases of  $\delta < 0$  and  $\delta > \frac{3}{2}$  the corresponding potentials both have maxima at  $\sigma = 0$ , due to a negative mass squared  $m_\sigma^2 < 0$ , which is in the origin of their instabilities. On the other hand, the cosmological models with  $0 < \delta \leq \frac{3}{2}$  have  $m_\sigma^2 \in [0, \frac{9}{4}]$ , and the scalar field potential (15) has a parabolic-like shape with a global minimum at  $\sigma = 0$ . Still, the models with  $\delta \in (0, \frac{3}{4})$  and  $\delta \in [\frac{3}{4}, \frac{3}{2}]$  represent two qualitatively different thermal histories of the universe evolution, reflecting the change in the TD properties of the fluid that occurs at  $\delta = \frac{3}{4}$ . Let us also remind that the restriction  $\delta \leq 1$  (as we have demonstrated above) is of a purely gravitational and hydrodynamical nature.

Our brief discussion of the thermodynamical and hydrodynamical properties of the self-dual cosmological models (8) has pointed out a set of arguments in favor of the physical consistency of the models with  $\frac{3}{4} \leq \delta \leq 1$ . It remains, however, to verify whether the conformal time SFD does act as thermodynamical UV/IR symmetry for these selected models.

## 2.2 SFD as UV/IR symmetry

The SFD transformations for the thermodynamical characteristics of modified Chaplygin gas follow the general equations (5). The constant entropy can be fixed by taking the radiation limit,<sup>4</sup> as

$$\tilde{S} = S = \frac{4}{3} V_0 (4\sigma_{sb} \rho_r^3)^{1/4},$$

and the “fixed point” of the physical volume transformation  $\tilde{V} = v_0^2/V$  is found to be  $v_0 \equiv V_0 (\rho_r/\rho_\Lambda)^{\frac{3}{4}}$ . Now the transformations of the temperature and of the entropy density  $s(a)$ , with the aid of Eqs.(9) and (3), can be written rather simply as

$$\tilde{T}(\tilde{a}) = \Omega^{1-2\delta} T(a), \quad \tilde{s}(\tilde{a}) = \frac{S^2}{v_0^2} \frac{1}{s(a)}, \quad \text{where } \Omega \equiv \frac{c_0^2}{a^2}. \quad (17)$$

Immediately we see that the fluid with  $\delta = 1/2$  has the interesting property of its temperature being invariant under the scale factor duality,  $\tilde{T}(\tilde{a}) = T(a)$ , and thus  $T_{in} = \infty = T_f$ . For all other values of  $\delta$ , the temperature transformation  $\tilde{T}(T)$  is given by

$$\left( \tilde{T} T \right)^{\frac{1}{4\delta-2}} = T_* \left( T^{\frac{2\delta}{2\delta-1}} + \tilde{T}^{\frac{2\delta}{2\delta-1}} \right)^{\frac{1-\delta}{\delta}}, \quad (18)$$

obtained by substituting the scale factor  $(a/c_0)^{2-4\delta} = \tilde{T}/T$  from Eqs.(17) into Eqs.(9). The *fixed point* of this transformation is given by the SFD invariant temperature

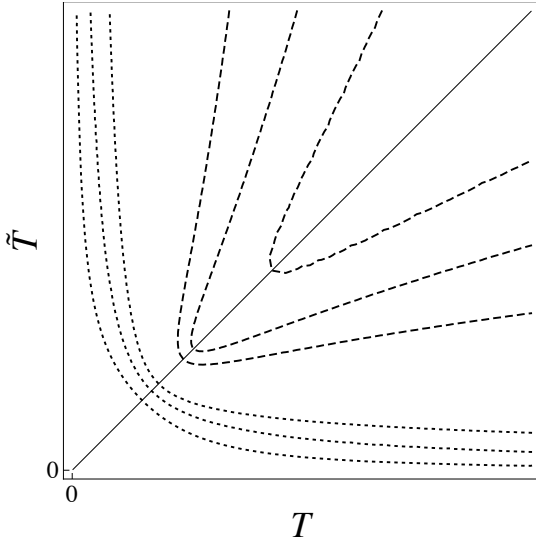
$$T_0 = T(\eta_c) = \tilde{T}(\eta_c) = 2^{\frac{1-\delta}{\delta}} T_* \quad (19)$$

of the fluid at the moment  $\eta_c$  of the transition from decelerated to accelerated expansion, i.e.  $T_0$  corresponds to  $q(\eta_c) = 0$ . At this same instant, the physical volume of the fluid reaches the invariant value  $v_0$ , and the EoS parameter takes the particular value  $\omega(\eta_c) = -\frac{1}{3}$ , corresponding to a cosmic string gas — this is the only value of  $\omega$  invariant under the last of Eqs.(3).

We next consider the conditions under which the above transformation  $\tilde{T}(T)$  acts as a high-to-low temperature UV/IR symmetry. Since (18) gives the explicit action of SFD on the temperature of the self-dual fluid, it maps ‘initial’ temperatures (at small values of  $a(\eta)$ ) to ‘final’ temperatures (at large values of  $a(\eta)$ ). But this evolution, as described in Sect.2.1, depends qualitatively on the values of  $\delta$  listed in cases (12), so the different behaviors must be described by (18). Although the latter is a complicated formula, the limit  $T \rightarrow \infty$  gives the most relevant information about the nature of the symmetry. For  $\delta > 1/2$  we find a simple asymptotic relation between the pair of dual temperatures,

$$\tilde{T} \approx T_* \left( \frac{T_*}{T} \right)^{\frac{1}{4\delta-3}}, \quad (20)$$

<sup>4</sup>I.e. when  $a^4 \ll \rho_r/\rho_\Lambda$ ; see footnote 2 above.



**Fig. 2** Dual temperatures related by Eq.(18), for  $\delta > 0$ . Dashed lines correspond to  $0 < \delta < \frac{3}{4}$  and dotted lines to  $\delta > \frac{3}{4}$ . The diagonal line corresponds to the limiting case  $\delta = \frac{1}{2}$ .

which confirms for  $\delta > \frac{3}{4}$  the expected mapping between  $T \approx \infty$  and  $\tilde{T} \approx 0$  and vice-versa. The corresponding monotonic curves of  $\tilde{T}(T)$  are depicted in Fig.2 as dotted lines. Note that in the particular case of  $\delta = 1$  both the asymptotic formula (20) and the exact one, Eq.(18), coincide:  $\tilde{T} = \frac{T^2}{T_*}$ . As anticipated in Sect.2.1, the  $\delta = \frac{3}{4}$  model represents a notable exception — the initial temperature  $T \approx \infty$  is now transformed into a *finite* final value  $\tilde{T} \approx T_*$ . The evolution of the temperature  $T(\rho)$  in the models with  $0 < \delta < \frac{3}{4}$  is *non-monotonic*, with a minimum value  $T_{cr}$ . Consequently,  $\tilde{T}(T)$  is not a single valued function, which gives the dashed lines in Fig.2. The “tip” of these curves (where they intersect the diagonal line) correspond to the invariant temperature (19),

$$T_0 = (2 - 2\delta)^{\frac{\delta-1}{\delta}} (3 - 4\delta)^{\frac{3-4\delta}{4\delta}} T_{cr},$$

cf. Eq.(10). As expected, since both initial and final temperatures diverge in this case, Eq.(18) indeed maps  $T \approx \infty$  to  $\tilde{T} \approx \infty$ . Finally, for  $\delta < 0$  the temperature is bounded,  $0 \leq T \leq T_{cr} \equiv T_{max}$ , and Eq.(20) is not valid. In this case we should consider instead the limit  $T \rightarrow 0$  of Eq.(18), and a similar asymptotic expansion of  $\tilde{T}(T)$  confirms that the SFD dual of  $T \approx 0$  is indeed  $\tilde{T} \approx 0$ , corresponding to the initial and final states being equally cold.

The results of our investigation of the SFD temperature  $\tilde{T}(T)$  transformations (18) may be summarized in the following *statement*: The relevant physical quantities,  $\rho(a)$ ,  $R(a)$ ,  $p(a)$  and  $T(a)$ , describing the evolu-

tion of the considered self-dual cosmological models (8) are, *all*, *monotonic functions* only for  $\delta \geq \frac{3}{4}$ . For these models, SFD does yield a “thermodynamical” UV/IR symmetry which maps high into low temperatures and pressures, and vice-versa. It is important to note that, independently of their complicated form (18), the  $\tilde{T}(T)$  transformations (for  $\delta \neq \frac{1}{2}$ ) can be identified as highly non-trivial realizations of the group  $Z_2$ , which contain the standard simple rule of inversion  $\tilde{T} = \frac{T^2}{T_*}$  as a special case corresponding to  $\delta = 1$ . The  $Z_2$  nature of the SFD temperature transformations becomes in fact evident if one instead consider their equivalent “Weyl form”, given by Eq. (17).

The above established restrictions on the validity of the thermodynamically extended UV/IR version of SFD symmetry, when combined with the consistency conditions derived in Sect.2.1, confirm the physical (and cosmological) relevance of a particular sub-family  $\delta \in [\frac{3}{4}, 1]$  of self-dual models (8).

### 3 Self-Duality of Apparent Horizons Thermodynamics

Asymptotically de Sitter universes develop a cosmological horizon with (asymptotically) constant radius. At a sufficiently late time the fluid inside any constant comoving volume  $V_0$  occupies a physical volume  $V = V_0 a^3$  whose radius is greater than the horizon, which leads to unavoidable causality inconsistencies. On the other hand, the horizons themselves are known to possess proper thermodynamical characteristics [22, 23]. We shall be considering here the *apparent horizon*,<sup>5</sup> whose physical radius  $\mathcal{R}_A$ , in flat FRW universes, is given simply by the Hubble radius:  $\mathcal{R}_A = 1/|H|$ . Their local definition makes them more appropriate for a thermodynamic description of the *observable universe* than the event horizons (see, e.g., [25] for a recent discussion).

The present section is, accordingly, devoted to the investigation of scale factor self-duality properties of the Cai-Kim’s apparent horizon thermodynamics [15]. The entropy and temperature prescribed to the apparent horizons are given by [15, 26]

$$T_A = \frac{1}{2\pi\mathcal{R}_A}, \quad S_A = \frac{1}{4G}(4\pi\mathcal{R}_A^2) = \frac{8\pi^2}{H^2}, \quad G \equiv \frac{1}{8\pi}. \quad (21)$$

While horizon entropy,  $S_A$ , follows the standard Bekenstein-Hawking “one fourth of area” rule [23], the particular choice of the temperature  $T_A$  is partially motivated

<sup>5</sup>In general, an apparent horizon may be defined as a marginally anti-trapped surface [24]. In the flat FRW universe this coincides with the Hubble radius where space-time expansion becomes superluminal and the expansion of inward radial null geodesics vanishes.

by the fact that, with Eq.(21), the Clausius relation  $dQ = T_A dS_A$  is equivalent to the Friedmann equations (2) on the horizon, corresponding to an almost adiabatic flux of energy (heat)  $dQ = -dE$  from the fluid through the apparent horizon. If matter satisfies the null energy condition (NEC),  $p + \rho \geq 0$ , then the 2<sup>nd</sup> TD law holds as well:

$$\frac{dS_A}{dt} = \frac{8\pi^2}{|H|^3} (p + \rho) \geq 0. \quad (22)$$

It is clear that, by construction, the SFD transformations of all the quantities characterizing the thermal history of the apparent horizon (APH) can be derived from the transformation of its radius:

$$\frac{\tilde{r}_A}{\tilde{a}} = \frac{r_A}{a}, \quad \text{or simply} \quad \tilde{r}_A = r_A, \quad (23)$$

where  $r_A \equiv \Upsilon_A/a$  is the comoving radius. They are, in fact, a direct consequence of the fluid density transformations (3), since the Friedmann Eq.(2) imposes the relation  $\Upsilon_A = \sqrt{\rho/3}$  between them. Then the APH entropy and temperature transformations

$$\tilde{a}\tilde{T}_A = aT_A, \quad \frac{\tilde{S}_A}{\tilde{a}^2} = \frac{S_A}{a^2}, \quad (24)$$

are obtained by substituting Eqs.(23) into (21). The comoving volume of the apparent horizon is not, of course, a constant. It is instructive to compare what happens with the entropy and energy of the fluid inside this changing volume, in contrast with the transformation (5) for constant  $V_0$ . The physical volumes,  $V_A = \frac{4}{3}\pi H^{-3}$  and  $V = V_0 a^3$ , transform quite differently:

$$\frac{\tilde{V}_A}{\tilde{a}^3} = \frac{V_A}{a^3}, \quad \tilde{V} = \frac{v_0^2}{V}. \quad (25)$$

But the internal energy  $E_f = \rho V_A$  and the entropy  $S_f = s V_A$  of the fluid inside the apparent horizon turn out to transform in the same way as in Eq.(5)

$$\tilde{S}_f = S_f, \quad \frac{\tilde{E}_f}{\tilde{a}} = \frac{E_f}{a}. \quad (26)$$

Recall that  $s = S/a^3$ . Now, because of the change of the comoving volume of the apparent horizon, the time evolution (obtained from Eqs.(2)) of the fluid entropies  $S_f = s V_A$  and  $S = s V_0 a^3$  manifests an important difference:

$$\frac{dS_f}{dt} = \frac{2\pi}{3} \frac{Sa}{(aH)^4} (\rho + 3p), \quad \text{versus} \quad \frac{dS}{dt} = 0. \quad (27)$$

Thus the 2<sup>nd</sup> TD law is *violated* for  $S_f$  during an accelerated expansion, when  $p < -\frac{1}{3}\rho$ . The simple expression<sup>6</sup>  $\rho = 12\pi^2 T_A^2$ , which is valid for every fluid, reveals another important feature of the APH thermodynamics, namely that  $T_A(a)$  and  $S_A(a)$  are always monotonic

functions. Notice the difference with the fluid's temperature (and pressure), whose (in general) non-monotonic behavior, as shown in Sect.2.1, causes certain TD instabilities.

When the universe is filled with the modified Chaplygin gas (8), the definitions (21) of horizon entropy and temperature naturally lead to the *self-duality* of APH thermodynamics under the thermodynamically extended SFD transformations:

$$\tilde{T}_A^{2\delta} - T_A^{2\delta} = \frac{T_A^{4\delta}}{T_A^{2\delta} - T_A^{2\delta}}, \quad \tilde{S}_A^\delta = S_A^\delta - S_A^\delta, \quad (28)$$

where  $T_A \equiv \frac{1}{2\pi} \sqrt{\frac{\rho_A}{3}}$  and  $S_A \equiv \frac{24\pi^2}{\rho_A}$ . As expected, the temperature is always a decreasing function of  $a \in (0, \infty)$

$$\infty > T_A \geq T_A \text{ for } \delta > 0, \text{ or } T_A \geq T_A \geq 0 \text{ for } \delta < 0, \quad (29)$$

and the APH entropy is an increasing one:

$$0 \leq S_A \leq S_A \text{ for } \delta > 0, \text{ or } S_A \leq S_A < \infty \text{ for } \delta < 0. \quad (30)$$

Observe that in the case of singular asymptotically de Sitter universes (i.e. for  $\delta > 0$ ) the *final* APH temperature  $T_A$  and entropy  $S_A$  are equal to the well known values of the de Sitter event horizon ones. In fact, as one can see from their definitions (21), the apparent horizon radius in the  $a \rightarrow \infty$  limit do coincide with the cosmological event horizon one. Instead, for  $\delta < 0$ , i.e. in the case of eternal universes, the same de Sitter values  $T_A$  and  $S_A$  represent now the *initial* APH temperature and entropy, obtained as  $a \rightarrow 0$ . In both cases, the SFD transformation laws (28) exchange the initial and final states temperatures and entropies, thus acting as a proper  $Z_2$  thermodynamical UV/IR symmetry, whose fixed points

$$T_{0A} = 2^{\frac{1}{\delta}} T_A, \quad S_{0A} = 2^{-\frac{1}{\delta}} S_A, \quad (31)$$

represent the values of APH temperature and entropy at the deceleration-to-acceleration transition, occurring at the instant of time  $\eta_c$ .

## 4 Concluding remarks

We shall complete our investigation with a few comments about the SFD features of the thermodynamics of the “*observable universe*”, i.e. thermodynamics of both the apparent horizon *and* the fluid contained inside its volume  $V_A$ , combined. One of the main challenges here is the validity of the generalized 2<sup>nd</sup> law, i.e. the growth of the *total* entropy  $S_t = S_f + S_A$ , where  $S_A$  is the area entropy of the horizon and  $S_f = S V_A$  the entropy of the fluid inside it. As can be seen from Eqs.(27) and (22), due to the decreasing of  $S_f$  in the accelerated expansion phase, it is rather difficult to construct

<sup>6</sup>It represents in fact the Friedmann equation, when the APH temperature definition (21) is taken into account.

a  $\Lambda$ CDM-like cosmology satisfying the generalized 2<sup>nd</sup> law during the entire evolution of the universe. It turns out, however, that the considered SFD symmetric models (8) do offer an example. It is the  $\delta = 3/4$  self-dual model, provided that, as  $a \rightarrow \infty$ , the asymptotic value  $S_t \rightarrow S_\Lambda$  obeys the bound

$$S_\Lambda \geq \frac{2\sigma_{sb}}{3\pi^2}, \quad \text{or equivalently} \quad \rho_\Lambda^{1/4} \leq \frac{2\pi\sqrt{3}}{(4\sigma_{sb})^{1/4}}, \quad (32)$$

as we have shown in ref.[14]. We should also recall that, apart of its simple TD properties described in Sect.2.1, this very special *SFD symmetric* cosmological model almost repeats the  $\Lambda$ CDM energy density evolution — it behaves as radiation at early times and its late time asymptotic is given by a cosmological constant accompanied by a cold dark matter.

We have shown that conformal time scale factor self-duality, when implemented as a symmetry principle, imposes a set of thermodynamical restrictions on the EoS parameters of the self-dual fluid and therefore may be used to select cosmological models.<sup>7</sup> Our final comment concerns the eventual applications of SFD *not* as a symmetry principle, but rather as an effective transformation relating large and small scales of a pair of *different* dual solutions to the Friedmann equations, each one with a *different* fluid. E.g., for  $(a, \rho, p)$  being given by flat  $\Lambda$ CDM with radiation, Eq.(3) gives as a dual the universe  $(\tilde{a}, \tilde{\rho}, \tilde{p})$  such that

$$\rho = \frac{\rho_r}{a^4} + \frac{\rho_d}{a^3} + \rho_\Lambda, \quad \tilde{\rho} = \frac{\tilde{\rho}_r}{\tilde{a}^4} + \frac{\tilde{\rho}_{dw}}{\tilde{a}} + \tilde{\rho}_\Lambda, \quad (33)$$

where  $\tilde{\rho}_{dw}/\tilde{a}$  represents the density of a domain walls gas, dual to the dust term  $\rho_d/a^3$ . The methods developed here for self-dual cosmologies can be easily extended to the thermodynamics of these pairs of SFD dual (but not *self*-dual) fluids. Thus one might employ the corresponding UV/IR *duality* transformations in the description of the high temperature behavior of a given cosmological model in terms of the low temperature data of its dual.

**Acknowledgments.** ALAL thanks CAPES (Brazil) for financial support. The research of UCdS is partially supported by FAPES (Brazil).

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<sup>7</sup>Including pre-big-bang models, where the slightly relaxed condition of “partial self-duality” is also useful [14].



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